8. Quantum Logic Gates and Universal Gate Sets

Chang Woo Kim
Computational Chemistry Group
Department of Chemistry, JNU

Quantum Logic Gates

Quantum logic gates are ways of unitarily manipulating the state of qubit(s).

Every quantum logic gate can be represented as a unitary matrix in an abstract level.

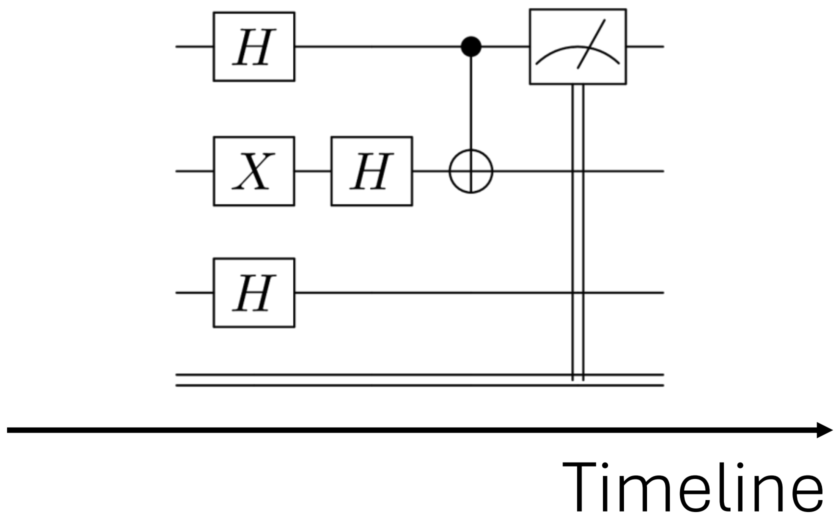
A single-qubit gate acts on one qubit and can be represented as a 2×2 matrix. In contrast, a multi-qubit or n-qubit gate is a $2^n \times 2^n$ matrix.

The actual implementation of quantum logic gates are different for every qubit platform, but one thing is for sure: realizing multi-qubit gates are far more demanding compared to single-qubit gates.

Quantum Circuits

To represent a quantum algorithm by using quantum logic gates, we put logic gates in order in the form of quantum circuits.

In the quantum circuits, each qubit is represented as a string along the time axis, and the gates are placed in a successive order to the right.



3 qubits

1 classical bit

Example of Single-Qubit Gates

Pauli gates

X: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Y: $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ Z: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Rotational gates

R_x R_y R_z

(Lecture 7, Slide 4 for the explicit representations)

Hadamard gate

H: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Phase gate

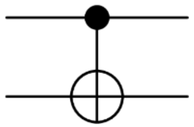
S: $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$\pi/8$ gate

T: $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

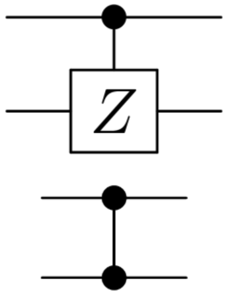
Example of Two-Qubit Gates

Controlled NOT (CNOT, CX) gate



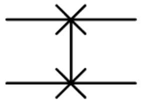
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Controlled Z (CZ) gate



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

SWAP gate



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\sqrt{\mathrm{SWAP}}$ gate

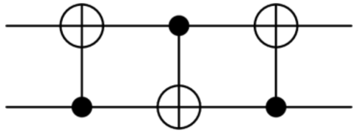
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

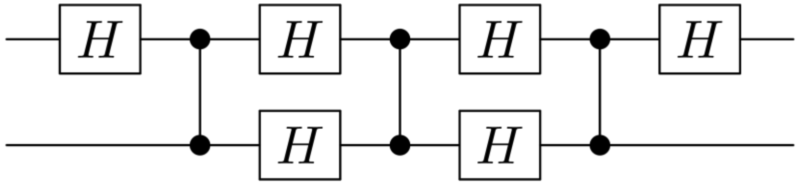
iSWAP gate

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

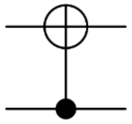
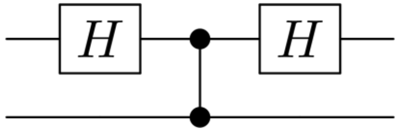
Universal Gate Set

It is known that any n-qubit logic gate can be expressed by using single-qubit gates and CNOT gates.

For example, a SWAP gate can be decomposed as  ,

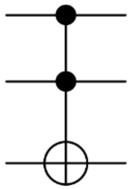
which can be represented as 

.

because  = 

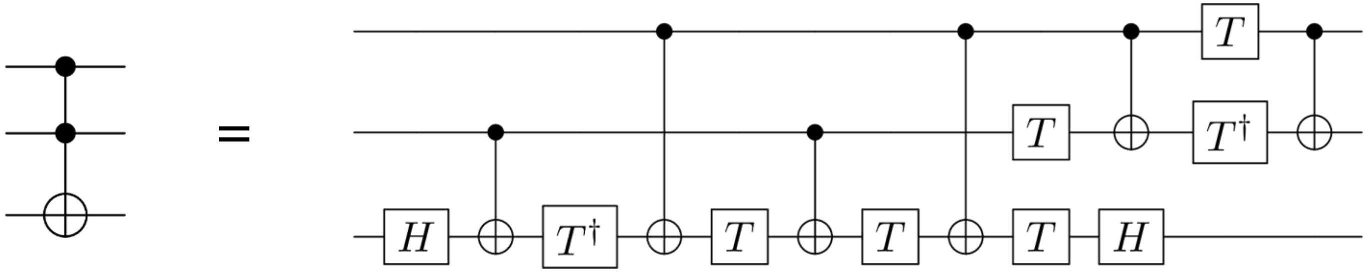
and the Hadamard gate is its own inverse.

Universal Gate Set

The Toffoli gate is symbolized as  , whose matrix form is

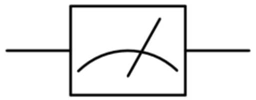
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Below is its decomposition into single-qubit gates and CNOT gates:



Measurement

At the end, the qubit state is measured and the outcome is converted into a classical bit. This is represented by the circuit diagram



For every measurement, the qubit state $|\psi\rangle$ is projected onto a fixed state $|p\rangle$ and the orthogonal state $|q\rangle$.

By repeating the measurement, we obtain the probabilities associated with the two states,

$$P_p = |\langle p|\psi\rangle|^2,$$
$$P_q = |\langle q|\psi\rangle|^2 = 1 - P_p.$$

Measurement

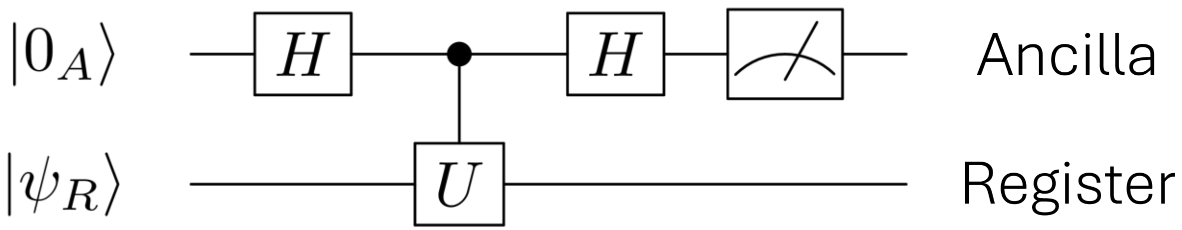
Suppose a qubit is at the state $|\psi\rangle = \alpha|0\rangle + \beta e^{i\theta}|1\rangle$ (α and β real), and we can only make a measurement in the logical basis $\{|0\rangle, |1\rangle\}$.

Question: how can we specify the qubit state? You can use any single-qubit gate you want.

Hadamard Test

An ancilla is an extra qubit that is used to indirectly capture information without performing measurements on the main “register” qubits.

Consider the circuit below:



Just before the measurement of the ancilla, the state of the qubits are

$$|\Psi\rangle = \frac{1}{2}(|0_A\rangle \otimes (\hat{I} + \hat{U})\,|\psi_R\rangle + |1_A\rangle \otimes (\hat{I} - \hat{U})\,|\psi_R\rangle),$$

whose associated probabilities are $P_0 = |\,\langle 0_A|\Psi\rangle\,|^2$ and $P_1 = |\,\langle 1_A|\Psi\rangle\,|^2$.

Hadamard Test

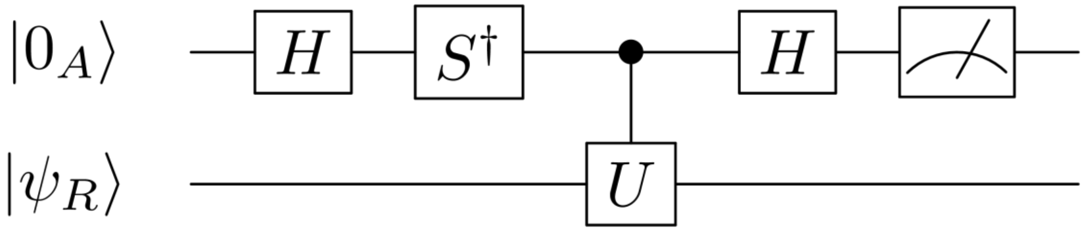
The probabilities are expressed as

$$P_0 = \frac{1 + \mathrm{Re}\,\langle\psi|\,\hat{U}\,|\psi\rangle}{2}, \quad P_1 = \frac{1 - \mathrm{Re}\,\langle\psi|\,\hat{U}\,|\psi\rangle}{2},$$

which satisfy

$$P_0 - P_1 = \mathrm{Re}\,\langle\psi|\,\hat{U}\,|\psi\rangle\,.$$

To obtain the imaginary part $\mathrm{Im}\,\langle\psi|\,\hat{U}\,|\psi\rangle$, we flip the phase of the ancilla qubit with an additional gate:



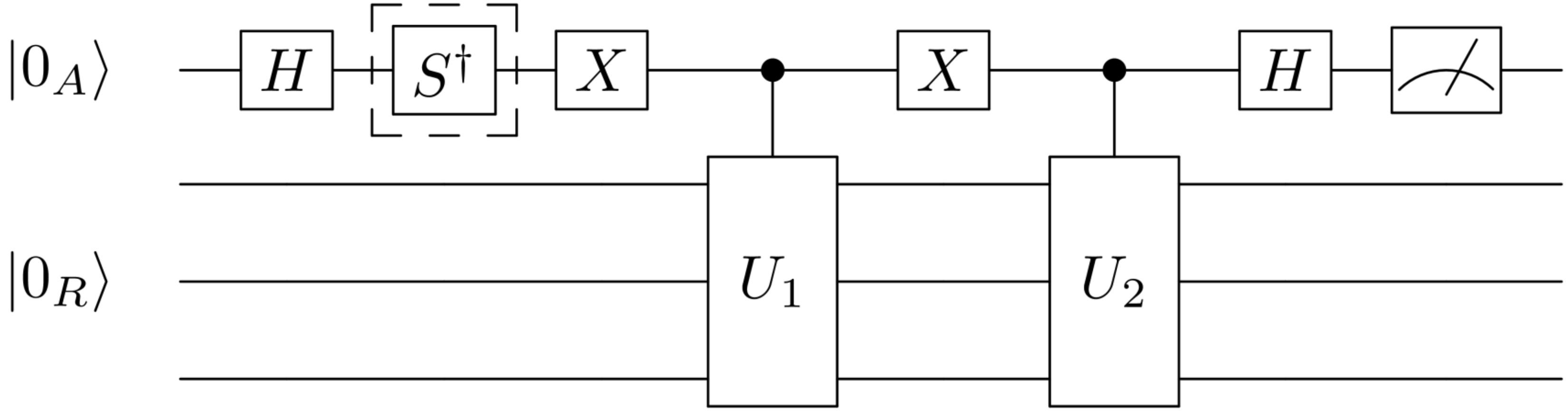
$$S^\dagger = \begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}$$

https://en.wikipedia.org/wiki/Hadamard_test

Inner Product between Two States

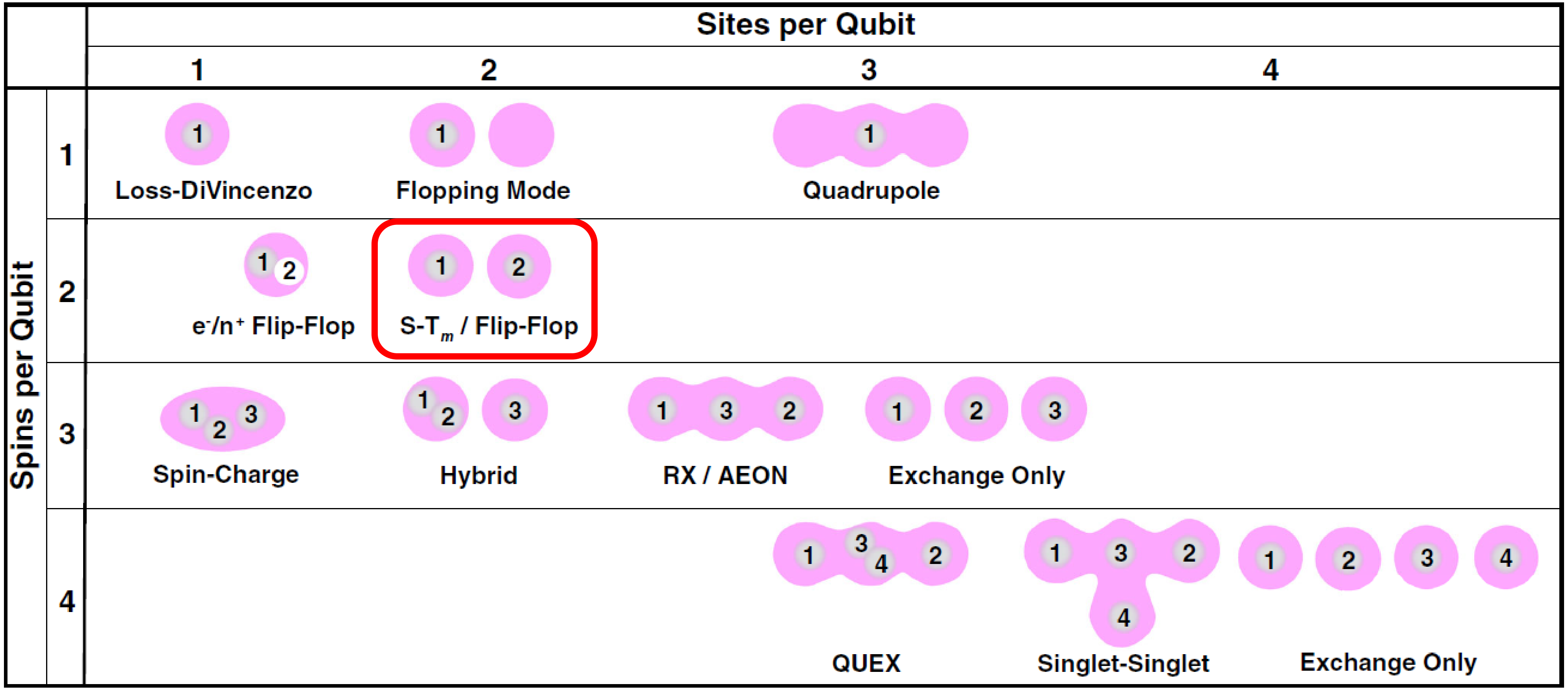
Suppose we have two different multi-qubit gates which act to the vacuum state as $\hat{U}_1 \left|0_R\right\rangle = \left|\psi_{R1}\right\rangle$ and $\hat{U}_2 \left|0_R\right\rangle = \left|\psi_{R2}\right\rangle$.

We can calculate the inner product between the two states $\langle\psi_{R1}|\psi_{R2}\rangle$ without directly gathering information about $\left|\psi_{R1}\right\rangle$ and $\left|\psi_{R2}\right\rangle$ by using the circuit below:



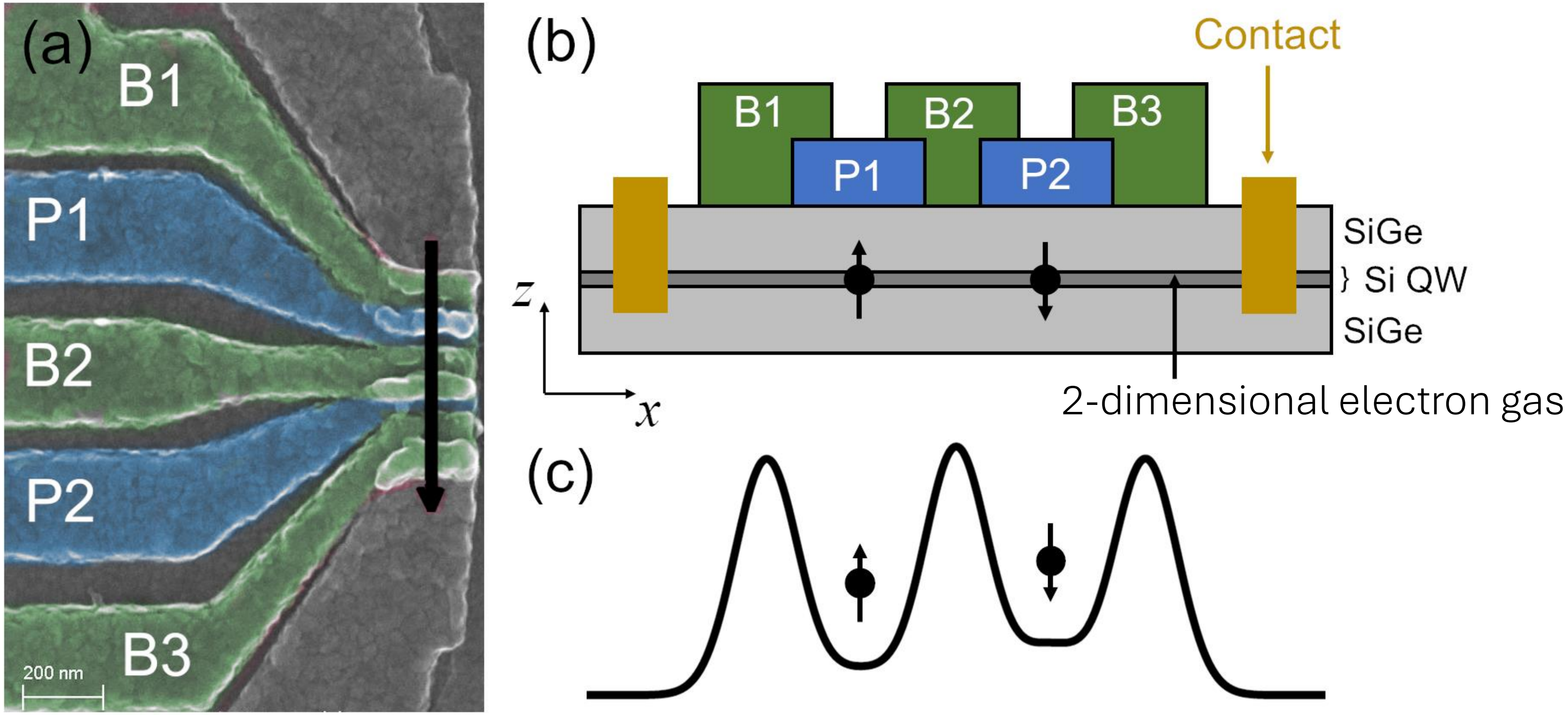
https://en.wikipedia.org/wiki/Hadamard_test

Semiconductor Quantum Dot Qubit



Burkard, G.; Ladd, T. D.; Pan, A.; Nichol, J. M.; Petta, J. R. Rev. Mod. Phys. 95, 025003 (2023)

Gate-Defined QDs

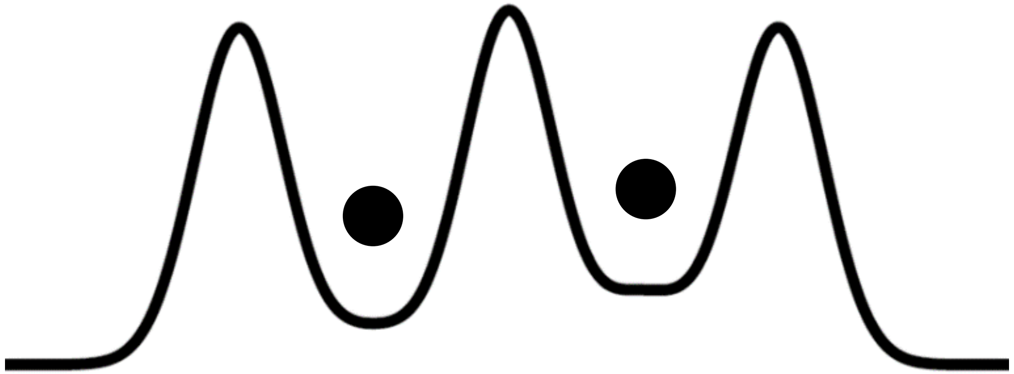


Electrostatic potential formed by electrode gates

<u>C. W. Kim*</u>, J. M. Nichol, A. N. Jordan, I. Franco*, *PRX Quantum* **2022**, *3*, 040308.

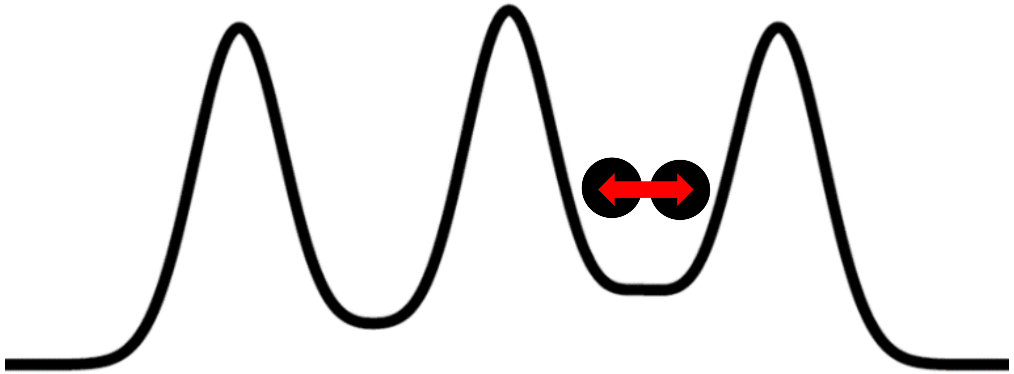
Semiconductor Quantum Dot Qubit

Consider two electrons trapped in a double quantum dot (DQD).



(1,1)
charge configuration

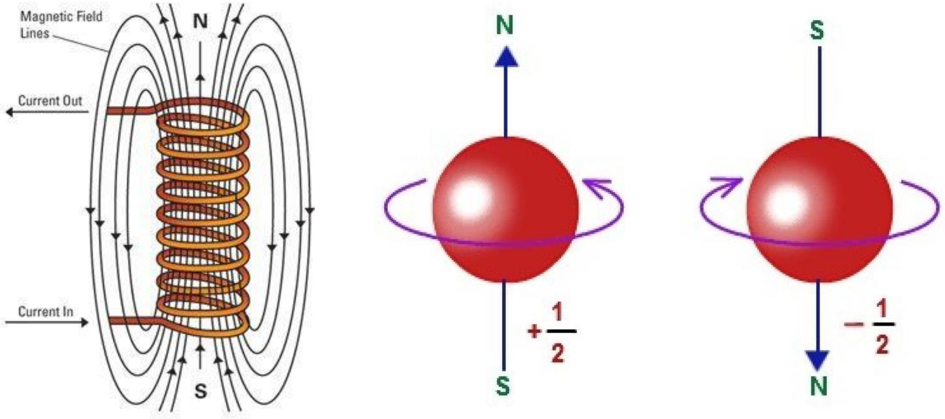
$$\epsilon_{(1,1)} \approx \epsilon_{\mathrm{L}} + \epsilon_{\mathrm{R}}$$



(0,2)
charge configuration

$$\epsilon_{(0,2)} \approx 2\epsilon_{\mathrm{R}} + \epsilon_{\mathrm{pair}}$$

Semiconductor Quantum Dot Qubit



Spin states of two electrons are classified into singlet and triplets:

$|\uparrow\uparrow\rangle \quad |\uparrow\downarrow\rangle$

$|\downarrow\uparrow\rangle \quad |\downarrow\downarrow\rangle$

Degenerate

External magnetic field

B_z

$$\Delta E = g\mu_{\mathrm{B}} B_z$$

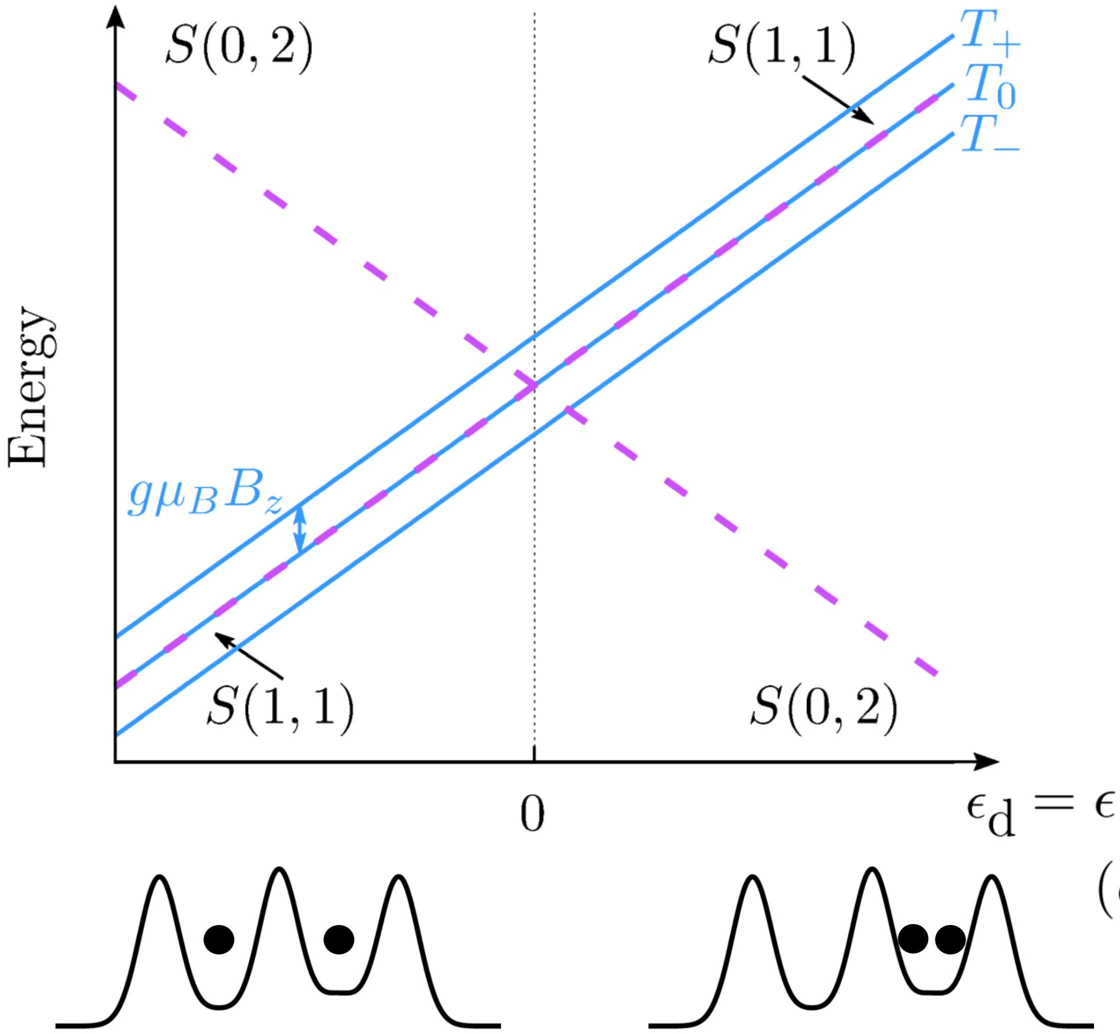
$$|T_+\rangle = |\uparrow\uparrow\rangle$$

$$|T_0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$|S\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$|T_-\rangle = |\downarrow\downarrow\rangle$$

Semiconductor Quantum Dot Qubit



$\epsilon_{\mathrm{d}} = \epsilon_{(1,1)} - \epsilon_{(0,2)}$

$(\epsilon_{\mathrm{L}} + \epsilon_{\mathrm{R}}) - (2\epsilon_{\mathrm{R}} + \epsilon_{\mathrm{pair}})$

Singlets

$$|S(1,1)\rangle = \frac{|\uparrow_{\mathrm{L}}\downarrow_{\mathrm{R}}\rangle - |\uparrow_{\mathrm{L}}\downarrow_{\mathrm{R}}\rangle}{\sqrt{2}}$$

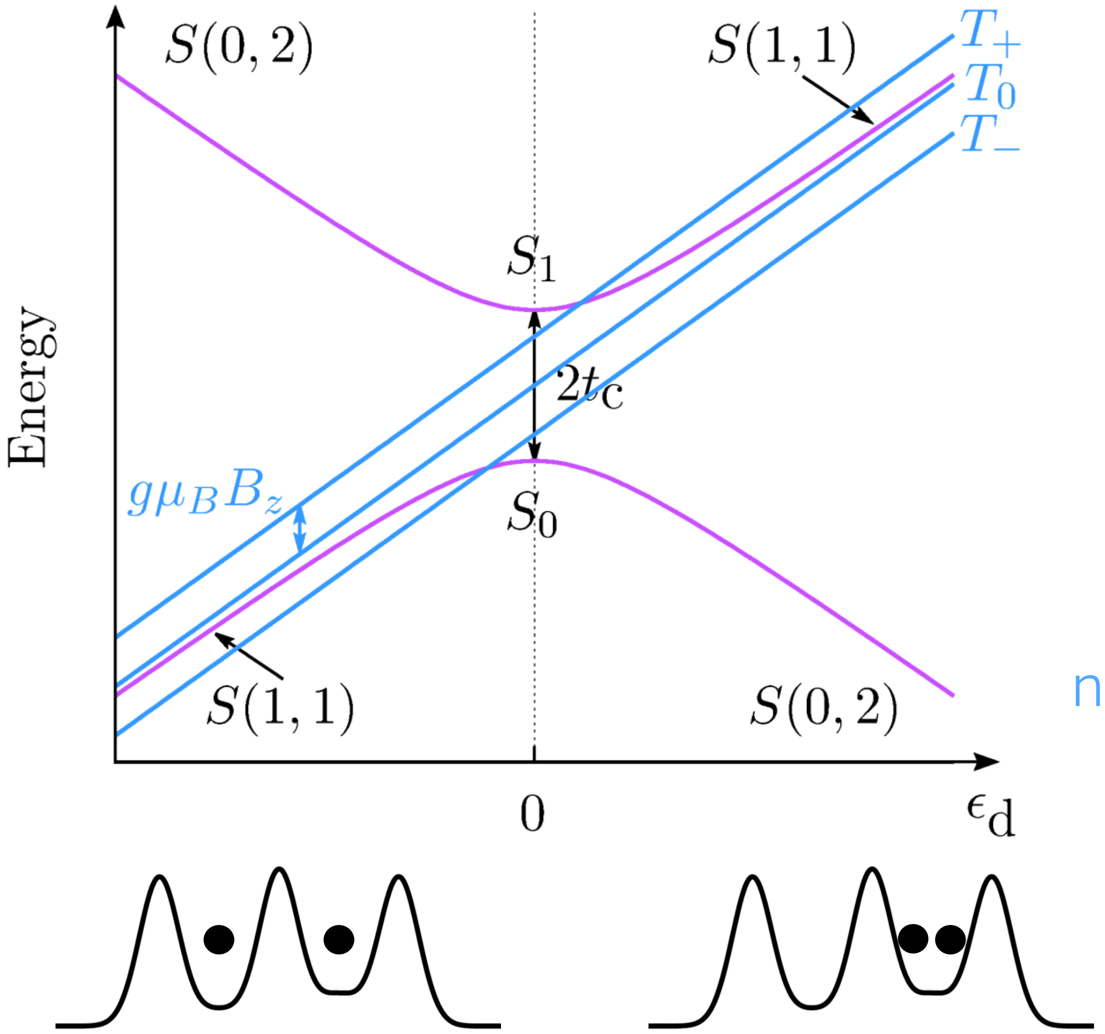
$$|S(0,2)\rangle = |\uparrow_{\mathrm{R}}\downarrow_{\mathrm{R}}\rangle$$

Triplets

$$|T_+\rangle = |\uparrow_{\mathrm{L}}\uparrow_{\mathrm{R}}\rangle \quad |T_-\rangle = |\downarrow_{\mathrm{L}}\downarrow_{\mathrm{R}}\rangle$$

$$|T_0\rangle = \frac{|\uparrow_{\mathrm{L}}\downarrow_{\mathrm{R}}\rangle + |\uparrow_{\mathrm{L}}\downarrow_{\mathrm{R}}\rangle}{\sqrt{2}}$$

Semiconductor Quantum Dot Qubit



Singlets

$$|S(1,1)\rangle = \frac{|\uparrow_L\downarrow_R\rangle - |\uparrow_L\downarrow_R\rangle}{\sqrt{2}}$$

$$|S(0,2)\rangle = |\uparrow_R\downarrow_R\rangle$$

Hybridize via tunnel coupling t_c to yield $|S_0\rangle$ and $|S_1\rangle$ (spin-charge hybridization)

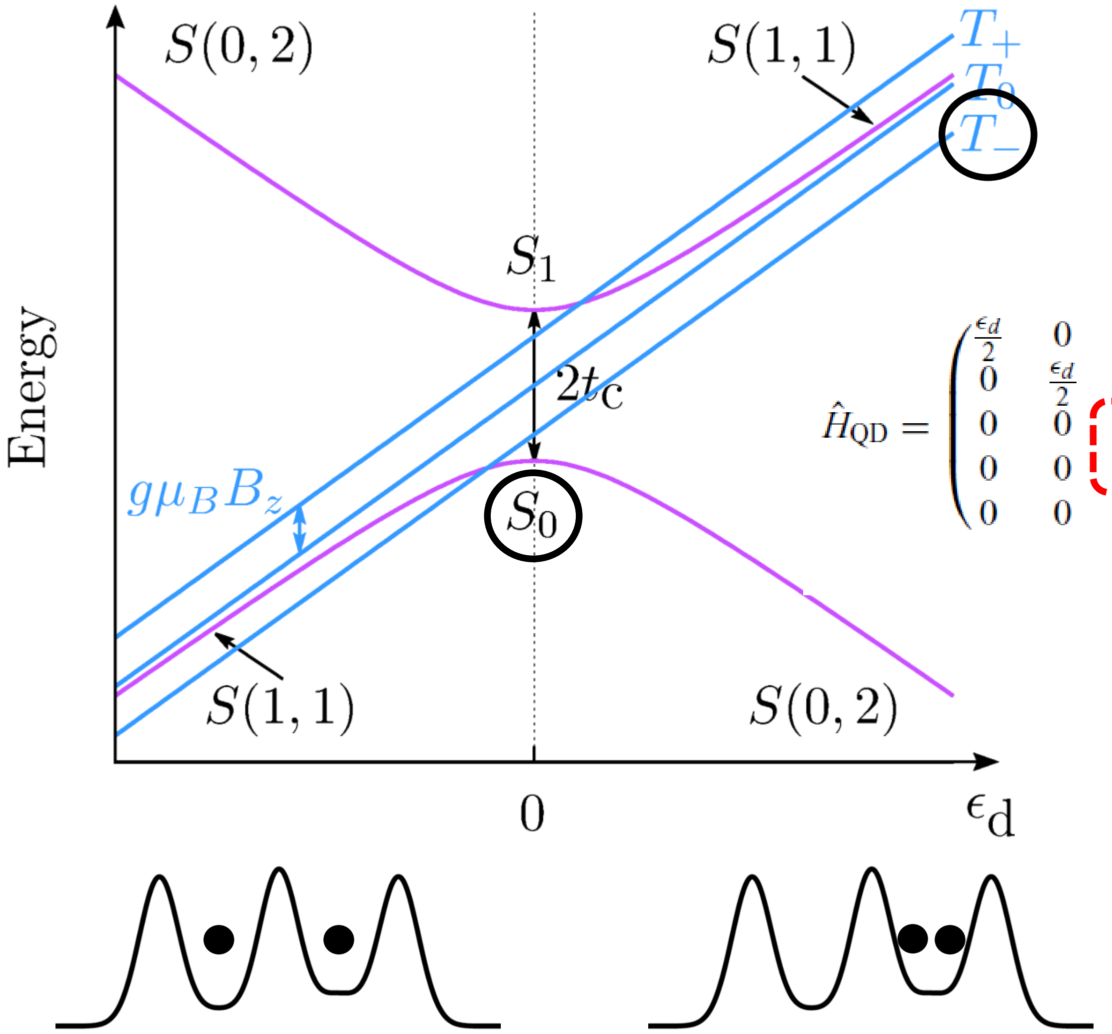
Triplets

no hybridization

$$|T_+\rangle = |\uparrow_L\uparrow_R\rangle \quad |T_-\rangle = |\downarrow_L\downarrow_R\rangle$$

$$|T_0\rangle = \frac{|\uparrow_L\downarrow_R\rangle + |\uparrow_L\downarrow_R\rangle}{\sqrt{2}}$$

Semiconductor Quantum Dot Qubit



$$\mathbf{B}_{\mathrm{L/R}} = \pm\frac{\Delta B}{2}\hat{x} - B_{\mathrm{avg}}\,\hat{z}$$

$$\hat{H}_{\mathrm{QD}} = \begin{pmatrix} \frac{\epsilon_d}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_d}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon_d}{2} & 0 & 0 \\ 0 & 0 & 0 & \epsilon_{S_0} & 0 \\ 0 & 0 & 0 & 0 & \epsilon_{S_1} \end{pmatrix} + g\mu_B \begin{pmatrix} B_{\mathrm{avg}} & 0 & 0 & -\frac{\Delta B}{2\sqrt{2}}\sin\theta & -\frac{\Delta B}{2\sqrt{2}}\cos\theta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -B_{\mathrm{avg}} & \frac{\Delta B}{2\sqrt{2}}\sin\theta & \frac{\Delta B}{2\sqrt{2}}\cos\theta \\ -\frac{\Delta B}{2\sqrt{2}}\sin\theta & 0 & \frac{\Delta B}{2\sqrt{2}}\sin\theta & 0 & 0 \\ -\frac{\Delta B}{2\sqrt{2}}\cos\theta & 0 & \frac{\Delta B}{2\sqrt{2}}\cos\theta & 0 & 0 \end{pmatrix}$$