7. Introduction to Qubits and Quantum Many-Body Systems

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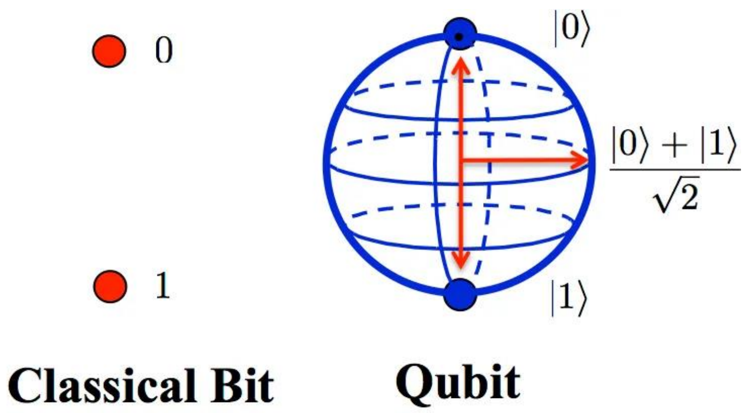
Qubits

A qubit is a controllable quantum two-state system which can be used to encode information.

As in the classical bit, we call each of the two states as $|0\rangle$ and $|1\rangle$.

Unlike classical bit which allows either $|0\rangle$ or $|1\rangle$, a qubit allows a superposition of $|0\rangle$ and $|1\rangle$.



https://medium.com/@adubey40/classical-bit-vs-qubit-fa6c6c06e8f

Bloch Sphere - Visualization of Qubit State

A quantum state of a two-level system can be written as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,$$

where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

We can always write $\alpha = |\alpha|e^{i\gamma}$, where γ is a real argument. Factoring the exponential leads to

$$|\psi\rangle = e^{i\gamma}(|\alpha|\,|0\rangle + |\beta|e^{i\phi}\,|1\rangle),$$

where the global phase factor $e^{i\gamma}$ does not affect the observable and therefore can be neglected. This allows us to write

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\,, \quad 0 \le \theta \le \pi \text{ and } 0 \le \phi \le 2\pi.$$

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This allow us to map any normalized $|\psi\rangle$ to a point on a sphere of unit radius in 3-dimensions. This sphere is called Bloch sphere.

Rotations w.r.t. the individual axes can be applied by

$$R_x(w) = \begin{pmatrix} \cos(w/2) & -i\sin(w/2) \\ -i\sin(w/2) & \cos(w/2) \end{pmatrix} = \exp(-iw\hat{\sigma}_x/2), \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$R_y(w) = \begin{pmatrix} \cos(w/2) & -\sin(w/2) \\ \sin(w/2) & \cos(w/2) \end{pmatrix} = \exp(-iw\hat{\sigma}_y/2), \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$R_z(w) = \begin{pmatrix} e^{-iw/2} & 0 \\ 0 & e^{iw/2} \end{pmatrix} = \exp(-iw\hat{\sigma}_z/2), \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Multiple Qubits

If we have two qubits, each qubit can have both $|0\rangle$ and $|1\rangle$ states, so that there are a total of four states: $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

For the rest of the courses, we will order the states of individual qubits from the rightmost position.

In general, n qubits can generate 2^n logical states.

Quantum states and operators for multiple qubits can be treated by using the concept of direct product, represented by the symbol $\otimes$.

If the first qubit is in the state $|\psi_1\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and the second qubit $|\psi_2\rangle = \gamma\,|0\rangle + \delta\,|1\rangle$, the combined multi-qubit state is represented as

$$|\Psi\rangle = |\psi_2\rangle \otimes |\psi_1\rangle = \alpha\gamma\,|00\rangle + \beta\gamma\,|01\rangle + \alpha\delta\,|10\rangle + \beta\delta\,|11\rangle\,.$$

Multiple Qubits

The matrix representations of operators can be also combined by direct product:

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} A_{11}\hat{B} & A_{12}\hat{B} & \cdots & A_{1m}\hat{B} \\ A_{21}\hat{B} & A_{22}\hat{B} & \cdots & A_{2m}\hat{B} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}\hat{B} & A_{m2}\hat{B} & \cdots & A_{mm}\hat{B} \end{pmatrix},$$

one can easily see that the direct product between $(m \times m)$ and $(n \times n)$ matrices is an $(mn \times mn)$ matrix.

When a single-qubit operator $\hat{A}$ acts on the first (second) qubit, its representation in the multi-qubit space is $\hat{I} \otimes \hat{A}$ ($\hat{A} \otimes \hat{I}$).

Multiple Qubits

Example: When the two qubits are in the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle , \qquad |\psi_2\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle ,$$

Calculate the expectation value of $\hat{\sigma}_x$ for both qubits, both in the separate spaces and the direct product space.

Entanglement

Not all multi-qubit states can be represented as a direct product between single-qubit states. Such states are called entangled states.

A good examples of entangled states are the so-called Bell states,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

The degree of entanglement varies among the quantum states, although there is no single agreed metric for quantifying the entanglement.

The entanglement can lead to many interesting consequences such as quantum teleportation.

Measurement of a Single Qubit

It is possible to measure a single qubit while preserving the superposition of other qubits.

Suppose we measured the j-th qubit of an n-qubit state $|\Psi(n)\rangle$ and obtained $|\phi_j\rangle$ as the outcome.

Then, the state of the remaining $n-1$ qubits after the measurement is

$$|\Psi'(n-1)\rangle = \frac{\langle\phi_j|\Psi(n)\rangle}{|\langle\phi_j|\Psi(n)\rangle|^2},$$

where the normalization factor $|\langle\phi_j|\Psi(n)\rangle|^2$ in the denominator is the probability of obtaining $|\phi_j\rangle$ from $|\Psi(n)\rangle$.

Measurement of a Single Qubit

For factorizable states, the measurement of one qubit does not affect the state of other qubit. In the case of the previous example,

$$|\psi_2\rangle \otimes |\psi_1\rangle$$
$$= \frac{1}{2\sqrt{2}}(i\,|00\rangle - |01\rangle + \sqrt{3}\,|10\rangle + \sqrt{3}i\,|11\rangle)$$

measurement of the 2nd qubit

$|0\rangle \otimes |\psi_1\rangle$ (Probability: 1/4)

$|1\rangle \otimes |\psi_1\rangle$ (Probability: 3/4)

On the other hand, an entangled state collapses to different states depending on the measurement outcome.

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

measurement of the 2nd qubit

$|0\rangle \otimes |1\rangle$ (Probability: 1/2)

$|1\rangle \otimes |0\rangle$ (Probability: 1/2)

Divincenzo's Criteria for Quantum Computer

As mentioned in the start of this lecture, the qubits must be controllable so that they can be used toward useful purposes.

In 2000, David P. DiVincenzo (1959~) proposed that a properly working quantum computer must:

- be a scalable physical system with well-characterized qubits,
- have the ability to be initialized to a simple fiducial state $|000\cdots\rangle$,
- have a universal set of quantum gates that can generate any combination of logical states,
- have long decoherence times compared to the gate-operation time,
- have capabilities for separately measuring its individual qubits.

Quantum Computing Platforms

There are many potential platforms for quantum computing:

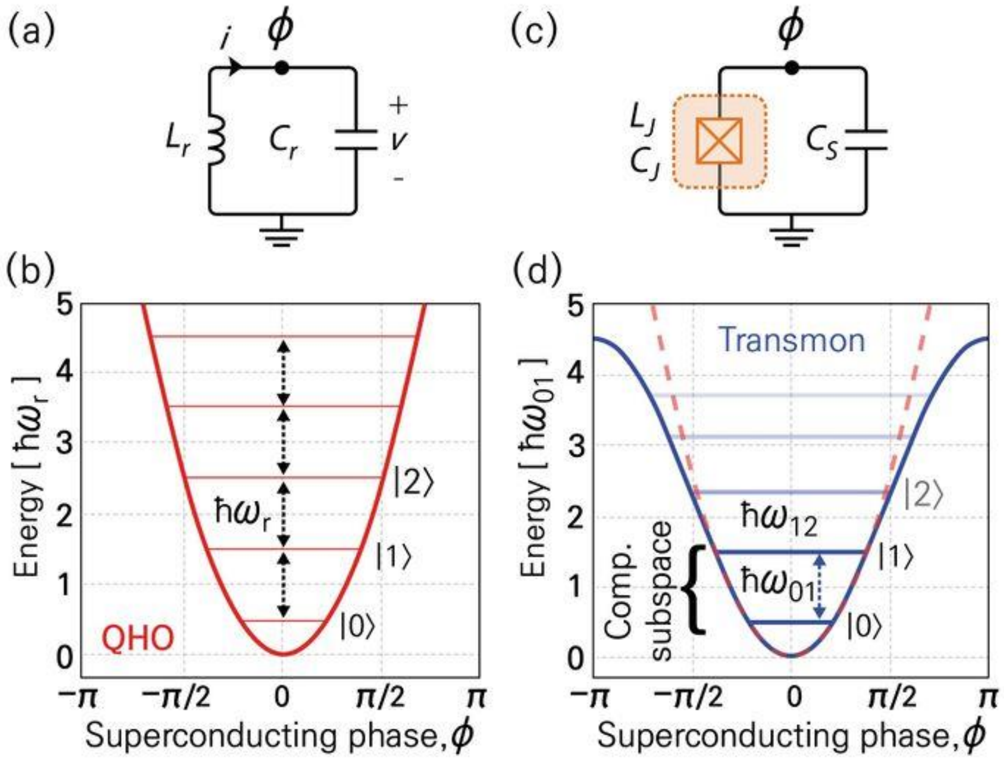
- Superconducting circuits
- Trapped Ions
- Entangled photons
- Electrons in quantum dots
- Color center in diamonds

Each platforms have its own advantages and disadvantages in:

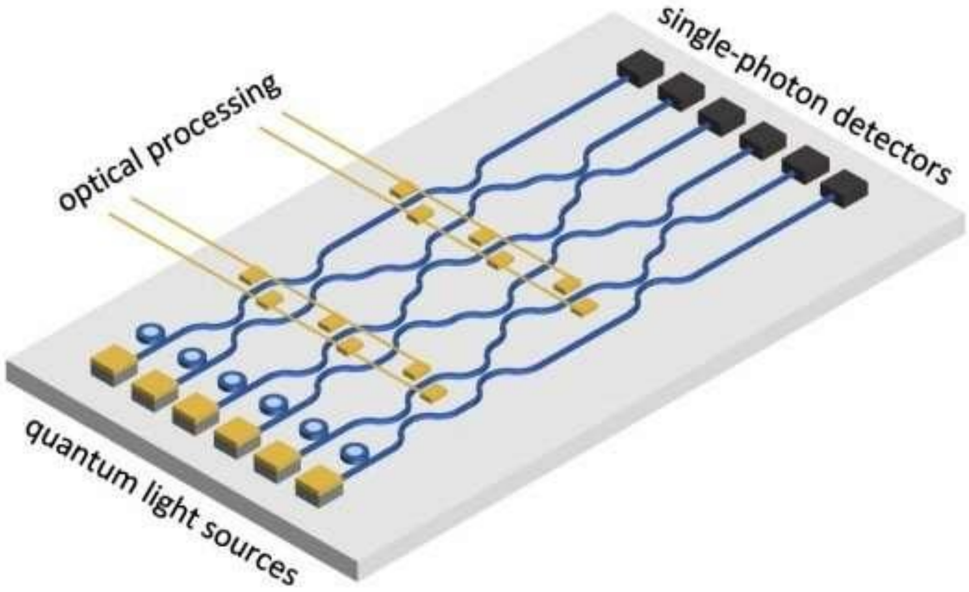
- Operating temperature
- Controllability
- Ratio between gate operation time and decoherence time

Superconducting Circuits / Entangled Photons

Superconducting circuits

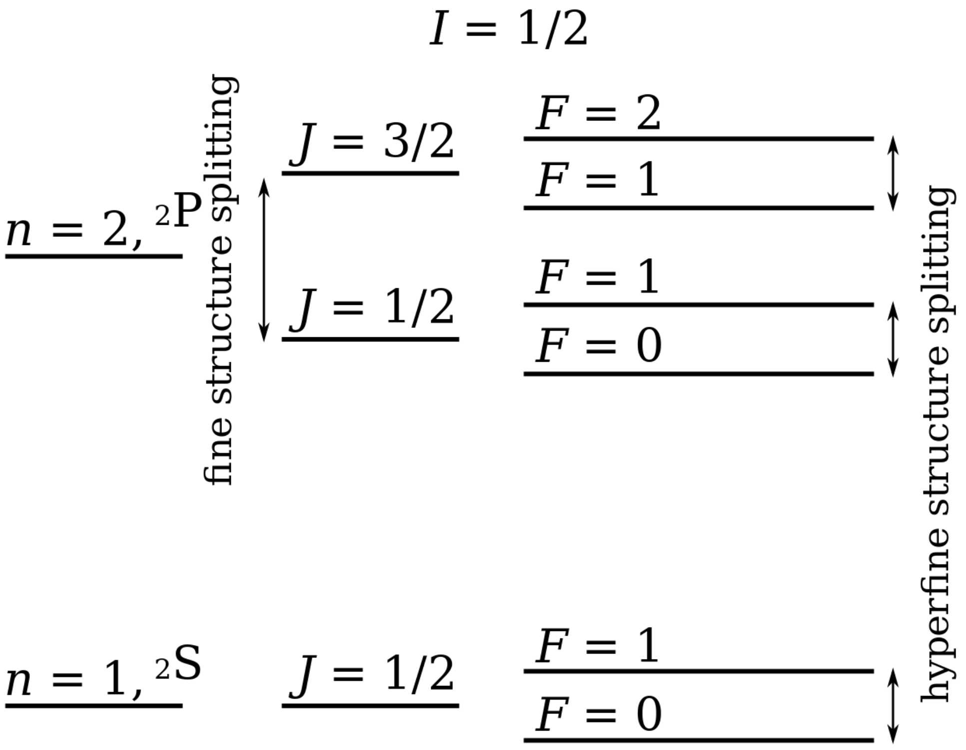


Entangled photons



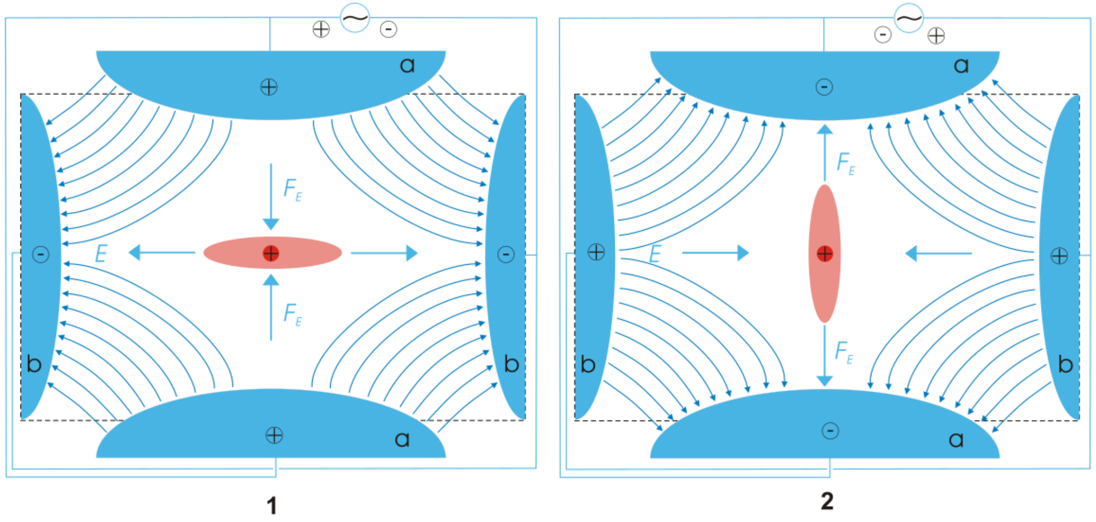
Krantz, P. et al. *Appl. Phys. Rev.* **6**, 021318, (2019).
https://phys.org/news/2019-10-quantum-photonics.html

Trapped Ions



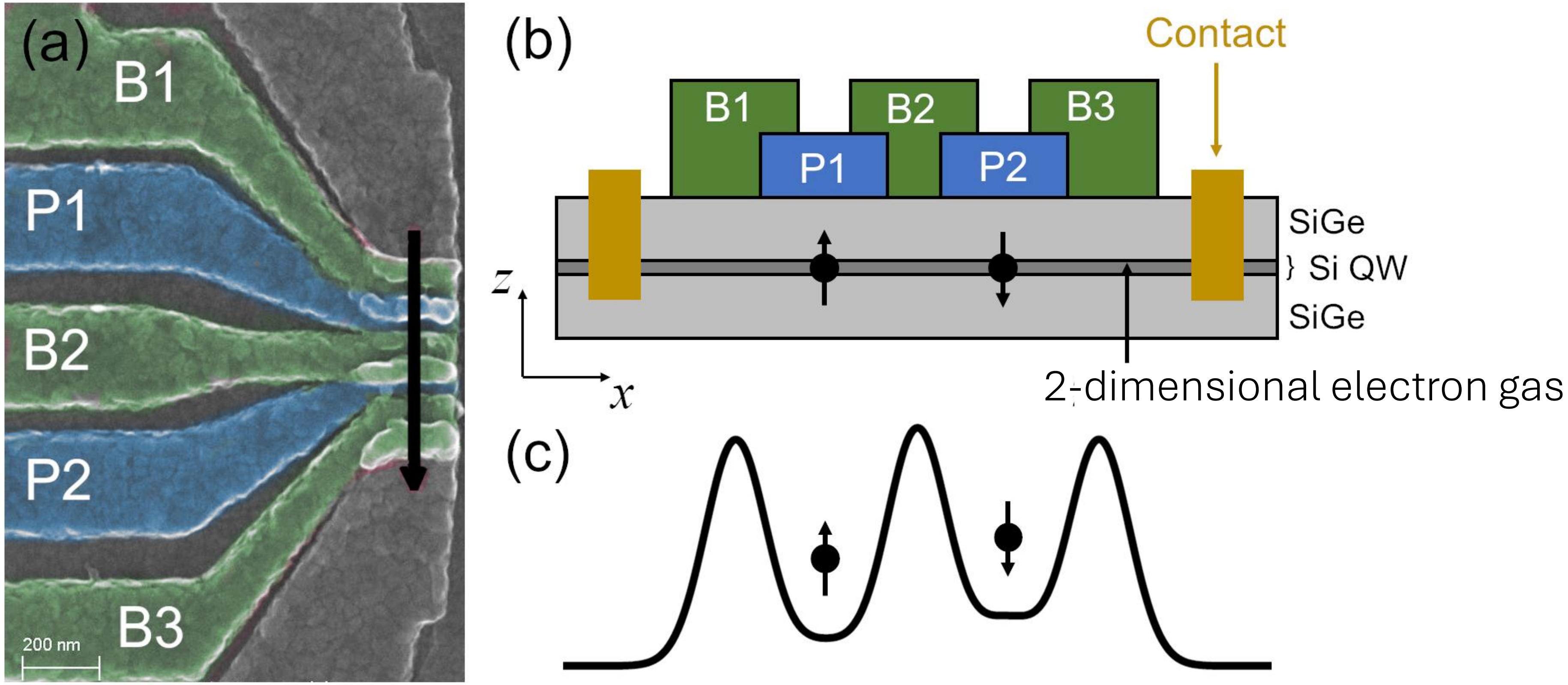
spin-orbit coupling of electrons

interaction between nucleus and orbiting electron



https://en.wikipedia.org/wiki/Hyperfine_structure
https://en.wikipedia.org/wiki/Quadrupole_ion_trap

Electrons in Quantum Dots



Electrostatic potential formed by electrode gates

Kim, C. W. and coworkers, *PRX Quantum* **2022**, *3*, 040308

Miscellaneous

A qudit is a generalization of a qubit to more than two levels.

A quantum emulator is a classical program which mimics the behavior of the real quantum computer.

The decoherence is a phenomenon that a quantum system loses its quantumness and returns to classical system.

A logical qubit is made of a group of physical qubits to implement error-correcting code that eliminates the error arising from the decoherence.

The quantum supremacy means the superiority of quantum computers over classical computers in certain algorithms.